An infinitary treatment of full μ-calculus

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Modal μ-calculus

Syntax of L_μ: $\bot \mid \top \mid p \mid \overline{p} \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \langle \mathfrak{a} \rangle \varphi \mid [\mathfrak{a}]\varphi \mid \mathsf{x} \mid \mu\mathsf{x}\,\varphi \mid \nu\mathsf{x}\,\varphi$
where $p \in \mathsf{Prop}$, $\mathfrak{a} \in \mathsf{Act}$, and $\mathsf{x} \in \mathsf{Var}$

Kozen's axiomatisation

- System K: PL + $\frac{\varphi}{[\mathfrak{a}]\varphi}$ + $[\mathfrak{a}](\varphi \to \psi) \to [\mathfrak{a}]\varphi \to [\mathfrak{a}]\psi$
- $\varphi(\mu\mathsf{x}\varphi(\mathsf{x})) \to \mu\mathsf{x}\varphi(\mathsf{x})$
- $\varphi(\psi) \to \psi \vdash \mu\mathsf{x}\varphi(\mathsf{x}) \to \psi$

Theorem (Kozen 1983; Walukiewicz 2000)

The proof system Koz is sound and complete for modal μ-calculus.

Techniques: *automata on infinite trees*, *games* and *tableaux constructions*

Full μ-calculus

Converse modalities:

- for each $\mathfrak{a} \in \mathrm{Act}$ we have $\bar{\mathfrak{a}} \in \mathrm{Act}$ and $\bar{\bar{\mathfrak{a}}} = \mathfrak{a}$
- Symmetric frames: $uR_{\mathfrak{a}}v$ iff $vR_{\bar{\mathfrak{a}}}u$
- So axiomatically, $\varphi \to [\bar{\mathfrak{a}}]\langle\mathfrak{a}\rangle\varphi$ for all every formula φ and $\mathfrak{a} \in \mathrm{Act}$

Comparisions

1. FMP fails: take e.g. $\nu x(\langle\mathfrak{a}\rangle x \wedge \mu y[\bar{\mathfrak{a}}]y))$
2. Satisfiability is decidable (Vardi 1998)
3. Is Koz + $p \to [\bar{\mathfrak{a}}]\langle\mathfrak{a}\rangle p$ complete for the full μ-calculus?

Axiomatisation for full μ-calculus

- Nested sequent: $\varphi_1, \varphi_2, \ldots, \varphi_k, \mathfrak{a}_1\{\Gamma_1\}, \mathfrak{a}_2\{\Gamma_2\}, \ldots \mathfrak{a}_l\{\Gamma_l\}$
- Intended reading of $\mathfrak{a}\{\Gamma\}$ is $i(\mathfrak{a}\{\Gamma\}) = [\mathfrak{a}] \bigvee_{\gamma \in \Gamma} i(\gamma)$.
- $L_\mu(\omega_1) = L_\mu \cup \{\nu^\alpha \mathsf{x}\varphi : \omega_1 > \alpha > 0\}$
- $\nu\mathsf{x}\varphi = \nu^{\omega_1}\mathsf{x}\varphi$.

$$\mathsf{Ax2}: \Gamma[p, \bar{p}]$$

$$\frac{\Gamma[\varphi, \psi]}{\Gamma[\varphi \vee \psi]}\vee \qquad \frac{\Gamma[\varphi] \qquad \Gamma[\psi]}{\Gamma[\varphi \wedge \psi]}\wedge \qquad \frac{\Gamma[\varphi(\mu\mathsf{x}\varphi)]}{\Gamma[\mu\mathsf{x}\varphi]}\mu$$

$$\frac{\Gamma[\mathfrak{a}\{\varphi\}]}{\Gamma[[\mathfrak{a}]\varphi]}[\mathfrak{a}] \qquad \frac{\Gamma[\mathfrak{a}\{\Delta, \varphi\}]}{\Gamma[\mathfrak{a}\{\Delta\}, \langle\mathfrak{a}\rangle\varphi]}\langle\mathfrak{a}\rangle \qquad \frac{\Gamma[\mathfrak{a}\{\Delta\}, \varphi]}{\Gamma[\mathfrak{a}\{\Delta, \langle\bar{\mathfrak{a}}\rangle\varphi\}]}\mathsf{con}$$

$$\frac{\Gamma[\varphi(\nu^\beta\mathsf{x}\varphi)] \quad \text{for all } \beta < \alpha}{\Gamma[\nu^\alpha\mathsf{x}\varphi]}\nu_\alpha$$

Figure: Axioms and rules of $K_\mu(\omega_1) + \mathsf{con}$

Saturation

Fix $\varrho \in L_\mu$, and let Γ & Δ be nested sequents.

Δ is a saturation of Γ if:

1. Δ *extends* Γ;
2. For all formulæ $\varphi, \psi \in L_\mu(\omega_1)$, $\mathfrak{a} \in \mathsf{Act}$ and nodes u, v in *tree*(Δ)
   a) $\varphi \wedge \psi \in u \Longrightarrow \varphi \in u$ or $\psi \in u$
   b) $\varphi \vee \psi \in u \Longrightarrow \varphi \in u$ and $\psi \in u$
   c) $[\mathfrak{a}]\varphi \in u \Longrightarrow \varphi \in v$ for some $\mathfrak{a}$-successor v of u
   d) if v is a $\mathfrak{a}$-successor of $u \Longrightarrow \{\varphi : \langle\mathfrak{a}\rangle\varphi \in u\} \subseteq v$ & $\{\varphi : \langle\bar{\mathfrak{a}}\rangle\varphi \in v\} \subseteq u$
   e) $\mu \mathsf{x}\varphi \in u \Longrightarrow \varphi(\mu\mathsf{x}\varphi) \in u$
   f) $\nu^\alpha \mathsf{x}\varphi \in u \Longrightarrow \varphi(\nu^\beta \mathsf{x}\varphi) \in u$ for some $\beta < \alpha$.

Lemma (Saturation Lemma)

If $\nvdash \Gamma$ there is a tree t_∞ extending Γ with the following properties

(i) *t_∞ satisfies all requirements of saturation i.e. a–f.*

(ii) *it is not the case that $p, \bar{p} \in u$ for any node u in t_∞*

Canonical model

Let $K = (W, \{R_{\mathfrak{a}}\}_{\mathfrak{a} \in \mathsf{Act}}, \lambda)$ be given by

- $W = \{u : u \text{ is a node in } t_\infty\}$
- $uR_{\mathfrak{a}}v$ iff $\{\varphi : \langle \mathfrak{a} \rangle \varphi \in u\} \subseteq v$ & $\{\varphi : \langle \bar{\mathfrak{a}} \rangle \varphi \in v\} \subseteq u$
- $\lambda(p) = \{u : p \notin u\}$

Remark: Nodes in t_∞ maybe labelled with an infinite set of formulae and be infinite branching. It is possible to finitise both.

Lemma (Truth Lemma)

For $u \in W$ and $L_\mu(\omega_1)$-formula φ

$$\varphi \in u \quad \Rightarrow \quad (K, u) \not\models \varphi$$

Theorem (Completeness)

The axiomatisation $K_\mu(\omega_1) + \mathsf{con}$ is complete for the full modal μ-calculus.

Bounding greatest fixed points

$$\frac{\Gamma[\varphi(\nu^{\beta}x\varphi)] \quad \text{for all } \beta < \omega_1}{\Gamma[\nu x\varphi]}\,\nu_{\omega_1} \quad \overset{?}{\rightsquigarrow} \quad \frac{\Gamma[\varphi(\nu^{n}x\varphi)] \quad \text{for all } n < \omega}{\Gamma[\nu x\varphi]}\,\nu_{\omega}$$

Theorem

- $K_\mu(\omega)$ *is sound for the modal* μ*-calculus — without appealing to FMP;*
- $K_\mu(\omega^\omega) + \mathsf{con}$ *is sound for the full* μ*-calculus;*
- $K_\mu(\kappa) + \mathsf{con}$ *is unsound for any* $\kappa < \omega^\omega$ *over symmetric frames.*

Strong closure

For a formula $\varrho \in L_\mu$ the $\mathbb{SC}(\varrho)$ is defined inductively:

- $\varrho \in \mathbb{SC}(\varrho)$
- If $\varphi \wedge \psi \in \mathbb{SC}(\varrho)$ or $\varphi \vee \psi \in \mathbb{SC}(\varrho)$ then $\varphi \in \mathbb{SC}(\varrho)$ and $\psi \in \mathbb{SC}(\varrho)$
- If $\langle \mathfrak{a} \rangle \varphi \in \mathbb{SC}(\varrho)$ or $[\mathfrak{a}]\varphi \in \mathbb{SC}(\varrho)$ then $\varphi \in \mathbb{SC}(\varrho)$
- If $\mu \mathsf{x} \varphi \in \mathbb{SC}(\varrho)$ then $\varphi(\bot) \in \mathbb{SC}(\varrho)$ and $\varphi(\mu \mathsf{x} \varphi) \in \mathbb{SC}(\varrho)$
- If $\nu^\alpha \mathsf{x} \varphi \in \mathbb{SC}(\varrho)$ then $\nu^\beta \mathsf{x} \varphi \in \mathbb{SC}(\varrho)$ and $\varphi(\nu^\beta \mathsf{x} \varphi) \in \mathbb{SC}(\varrho)$ $\forall \beta < \alpha$.

Proposition

Let $\varrho \in L_\mu$. *There is an ordering* $\sqsubseteq$ *such that*

1. $(\mathbb{SC}(\varrho), \sqsubseteq)$ *is a WQO.*
2. *If* $\varphi \sqsubseteq \psi$ *then* $rank(\varphi) \leq rank(\psi)$
3. *If* $\varphi \sqsubseteq \psi$ *then* $\|\psi\| \subseteq \|\varphi\|$

$(Q, \sqsubseteq)$ is a well-quasi order (WQO) if

- $\sqsubseteq$ is a reflexive and transitive relation on Q, and
- every infinite seq. $q_1, q_2, \ldots$ has an increasing pair $q_i \sqsubseteq q_j$ with $i < j$.

Definition (Smyth powerdomain)

For sets $X, Y \subseteq Q$ define $X \sqsubseteq Y$ iff $\forall \varphi \in Y \exists \psi \in X$ s.t. $\psi \sqsubseteq \varphi$.

Proposition

$\left(\mathcal{P}(\mathbb{SC}(\varrho)), \sqsubseteq\right)$ *is a wqo.*

Finitising/regularising the canonical model

- Along every infinite path in t_∞ there exists $uR^* v$ s.t. $u \sqsubseteq v$
- Delete the path from v onward and connect its predecessor to u
- Check truth lemma obtains

Theorem

$K_\mu(\omega)$ is sound and complete for Kripke frames.

For converse:

- A refinement of $\sqsubseteq$:

$$u \sqsubseteq_* v \text{ iff } v = f(u) \text{ for an order-preserving } f : On \longrightarrow On.$$

- For sets:

$$X \sqsubseteq_* Y \text{ iff } f(X)^\uparrow = Y^\uparrow \text{ for some increasing, order-preserving } f$$

Theorem

$K_\mu(\omega^\omega)$ is sound and complete for symmetric frames.

Unsound inferences

$$\frac{\Gamma[\varphi(\nu^\alpha \mathrm{x}\varphi)] \quad \text{for all } \alpha < \kappa}{\Gamma[\nu \mathrm{x}\varphi]} \nu_\kappa$$

- ν_ω is sound over arbitrary frames.
- ν_ω is not sound over symmetric frames:

$$\rho = [\bar{\mathfrak{a}}]\bot \quad \text{and} \quad \varphi(x) = (\rho \wedge \mu y \langle \mathfrak{a} \rangle (y \vee x)) \vee \langle \bar{\mathfrak{a}} \rangle (x \wedge \neg \rho)$$

  $\rho \longrightarrow \nu^n x\, \varphi$ holds in all infinite models, but $\rho \longrightarrow \nu^\omega x\, \varphi$ does not.
- ν_α is unsound for each $\alpha < \omega^2$, and unsound for each $\alpha < \omega^\omega$ over trees.
- ν_{ω^ω} is sound over symmetric frames.

Discussion

1. Generalisation to guarded fixed point logic
2. Trade-off between soundness and completeness
3. Open: Is Koz + con complete for symmetric frames?
4. Potential applications of the wqo $\sqsubseteq$ to other open problems in μ-calculus