

# Normalization by Evaluation for Intuitionistic Propositional Logic

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## 1 Intuitionistic Propositional Logic (IPL)

*Formulas* and hypotheses lists (contexts).

Atom	$\ni P, Q$		atomic propositions
Form	$\ni A, B, C, \dots$	$::= \top \mid A \wedge B \mid A \Rightarrow B$ $\mid \perp \mid A \vee B \mid P$	negative formulas positive formulas
Cxt	$\ni \Gamma, \Delta$	$::= \emptyset$ $\mid \Gamma.A$	empty context context extension

We write just  $A$  for the singleton context  $\emptyset.A$ .

Indices  $x$  into the context  $\Gamma$  of hypothesis, locating hypothesis  $A$ , are written  $x : \text{Hyp}(A)_\Gamma$  and defined inductively by the following constructors:

$$\frac{}{\text{top} : \text{Hyp}(A)_{\Gamma.A}} \quad \frac{x : \text{Hyp}(A)_\Gamma}{\text{pop}_B x : \text{Hyp}(A)_{\Gamma.B}}$$

Derivations  $t$  of the truth of a formula  $A$  under assumptions  $\Gamma$  (judgement  $\Gamma \vdash A$ ) are written  $t : \text{Tm}(A)_\Gamma$  and defined inductively as follows:

*Implication and hypotheses.*

$$\frac{x : \text{Hyp}(A)_\Gamma}{\text{hyp } x : \text{Tm}(A)_\Gamma} \quad \frac{t : \text{Tm}(B)_{\Gamma.A}}{\text{impl } t : \text{Tm}(A \Rightarrow B)_\Gamma} \quad \frac{t : \text{Tm}(A \Rightarrow B)_\Gamma \quad u : \text{Tm}(A)_\Gamma}{\text{impE } t u : \text{Tm}(B)_\Gamma}$$

*Truth and conjunction.*

$$\frac{}{\text{truel} : \text{Tm}(\top)_\Gamma} \quad \frac{t : \text{Tm}(A)_\Gamma \quad u : \text{Tm}(B)_\Gamma}{\text{andI } t u : \text{Tm}(A \wedge B)_\Gamma}$$

$$\frac{t : \text{Tm}(A \wedge B)_\Gamma}{\text{andE}_1 t : \text{Tm}(A)_\Gamma} \quad \frac{t : \text{Tm}(A \wedge B)_\Gamma}{\text{andE}_2 t : \text{Tm}(B)_\Gamma}$$

*Absurdity and disjunction.*

$$\frac{t : \text{Tm}(A)_\Gamma}{\text{orl}_1 t : \text{Tm}(A \vee B)_\Gamma} \quad \frac{t : \text{Tm}(B)_\Gamma}{\text{orl}_2 t : \text{Tm}(A \vee B)_\Gamma}$$

$$\frac{t : \text{Tm}(\perp)_\Gamma}{\text{falseE } t : \text{Tm}(C)_\Gamma} \quad \frac{t : \text{Tm}(A \vee B)_\Gamma \quad u : \text{Tm}(C)_{\Gamma.A} \quad v : \text{Tm}(C)_{\Gamma.B}}{\text{orE } t u v : \text{Tm}(C)_\Gamma}$$

## 2 Subformula property and normal derivations

For every derivable judgement  $\Gamma \vdash A$  there are infinitely many derivations  $t : \text{Tm}(A)_\Gamma$ , since arbitrary detours are allowed. For instance to prove  $\top$  we can also proceed very indirectly by introducing a hypothesis  $\top$  and then eliminating it by proof `trueE`:

$$\text{impE (impl (hyp top)) trueE} : \text{Tm}(\top)_\emptyset.$$

A sensible restriction is that the proof rules guarantee that only *subformulas* of  $A$  and the hypotheses in  $\Gamma$  are mentioned when deriving  $\Gamma \vdash A$ .

The following indexed grammar for *normal* derivations  $\text{Nf}(A)_\Gamma$  of judgement  $\Gamma \vdash A$  ensures the subformula property. It is defined mutually with a grammar  $\text{Ne}(A)_\Gamma$  of *neutral* derivations. Neutral derivations are “straight-line” consequences of hypotheses (without case distinction or absurdity elimination).

*Implication and hypotheses.*

$$\frac{x : \text{Hyp}(A)_\Gamma}{\text{hyp } x : \text{Ne}(A)_\Gamma} \quad \frac{t : \text{Nf}(B)_{\Gamma.A}}{\text{impl } t : \text{Nf}(A \Rightarrow B)_\Gamma} \quad \frac{t : \text{Ne}(A \Rightarrow B)_\Gamma \quad u : \text{Nf}(A)_\Gamma}{\text{impE } t u : \text{Ne}(B)_\Gamma}$$

*Truth and conjunction.*

$$\frac{}{\text{trueE} : \text{Nf}(\top)_\Gamma} \quad \frac{t : \text{Nf}(A)_\Gamma \quad u : \text{Nf}(B)_\Gamma}{\text{andI } t u : \text{Nf}(A \wedge B)_\Gamma}$$

$$\frac{t : \text{Ne}(A \wedge B)_\Gamma}{\text{andE}_1 t : \text{Ne}(A)_\Gamma} \quad \frac{t : \text{Ne}(A \wedge B)_\Gamma}{\text{andE}_2 t : \text{Ne}(B)_\Gamma}$$

*Absurdity and disjunction.*

$$\frac{t : \text{Nf}(A)_\Gamma}{\text{orl}_1 t : \text{Nf}(A \vee B)_\Gamma} \quad \frac{t : \text{Nf}(B)_\Gamma}{\text{orl}_2 t : \text{Nf}(A \vee B)_\Gamma}$$

$$\frac{t : \text{Ne}(\perp)_\Gamma}{\text{falseE } t : \text{Nf}(C)_\Gamma} \quad \frac{t : \text{Ne}(A \vee B)_\Gamma \quad u : \text{Nf}(C)_{\Gamma.A} \quad v : \text{Nf}(C)_{\Gamma.B}}{\text{orE } t u v : \text{Nf}(C)_\Gamma}$$

*Embedding.*

$$\frac{t : \text{Ne}(P)_\Gamma}{\text{ne } t : \text{Nf}(P)_\Gamma}$$

The introduction rules generate normal forms (Nfs) from Nfs. The elimination rules for negative formulas generate neutrals (Nes) from Nes, where `impE` requires a side argument in normal form.

Some care is needed with the elimination rules for positive formulas which are **falseE** and **orE**. These prove an arbitrary formula  $C$ , which, for the sake of the subformula property, should not be subject to further elimination but directly prove the goal. Thus, in addition to restriction the elimination to neutrals (perform case distinction only on neutrals), we require them to produce a normal form in case of **orE** from normal branches.

Any neutral derivation at atomic proposition  $P$  is considered normal. The restriction on atoms is not needed for the subformula property, but forces derivations to be  $\eta$ -long. We could drop the restriction, but even with this restriction, the calculus is complete, meaning that for every derivation (**Tm**) there exists a normal derivation (**Nf**). This statement is called *normalization* and we will prove it constructively in the following.

### 3 Categories and Presheaves

We define order-preserving embeddings (OPEs)  $\tau : (\Delta \leq \Gamma)$  of a context  $\Gamma$  into a larger context  $\Delta$  inductively by the following constructors.

$$\overline{\text{id}_\Gamma : \Gamma \leq \Gamma} \quad \overline{\tau : \Delta \leq \Gamma} \quad \overline{\tau : \Delta \leq \Gamma}$$

$$\text{id}_\Gamma : \Gamma \leq \Gamma \quad \text{weak}_A \tau : \Delta.A \leq \Gamma \quad \text{lift}_A \tau : \Delta.A \leq \Gamma.A$$

For example  $\text{weak}_D (\text{lift}_C (\text{weak}_B \text{id}_A)) : A.B.C.D \leq A.C$ . OPEs let us introduce extra, unused hypotheses into the context. Derivations of  $\Gamma \vdash A$  can be weakened to derivations  $\Delta \vdash A$  for  $\Delta \leq \Gamma$ .

OPEs are closed under composition: If  $\tau : \Delta \leq \Gamma$  and  $\tau' : \Phi \leq \Delta$  then  $\tau \circ \tau' : \Phi \leq \Gamma$ . Let **OPE** be the category of contexts (as objects) and OPEs (as morphisms).

**Exercise 1** Define composition of OPEs by induction and show that **OPE** is indeed a category.

The context-indexed sets  $\text{Hyp}(A)$ ,  $\text{Tm}(A)$ ,  $\text{Nf}(A)$ , and  $\text{Ne}(A)$  are closed under weakening. E. g. **Hyp**: for each  $\tau : \Delta \leq \Gamma$  we have a function

$$\text{Hyp}(A)_\tau : \text{Hyp}(A)_\Gamma \rightarrow \text{Hyp}(A)_\Delta,$$

i. e., a morphism in the category **SET** of sets and functions. The collection  $\text{Hyp}(A) : (\tau : \Delta \leq \Gamma) \rightarrow (\text{Hyp}(A)_\Gamma \rightarrow \text{Hyp}(A)_\Delta)$  of these function constitutes a contravariant functor  $\text{Hyp}(A) : \text{OPE} \rightarrow \text{SET}$  mapping OPEs to functions. In other words,  $\text{Hyp}(A)$  is a *presheaf* over **OPE** for each  $A$ , and so are  $\text{Tm}(A)$ ,  $\text{Nf}(A)$ , and  $\text{Ne}(A)$ .

**Exercise 2** Prove that  $\text{Hyp}(A)$ ,  $\text{Tm}(A)$ ,  $\text{Nf}(A)$ , and  $\text{Ne}(A)$  are presheaves over **OPE** by defining the map functions and proving the functor laws!

A function  $f$  between presheaves  $\mathcal{A}$  and  $\mathcal{B}$  is defined pointwise; we write  $f : \mathcal{A} \rightarrow \mathcal{B}$  for the **Cxt**-indexed collection

$$f_\Gamma : \mathcal{A}_\Gamma \rightarrow \mathcal{B}_\Gamma$$

of functions (morphisms in **SET**). Presheaves and functions between them make a category by virtue of pointwise identity and composition. However, typically

the functions we consider are *natural* in the context index, i. e., they are *natural transformations* between presheaves, which means that they commute with OPEs. Given  $\tau : \Delta \leq \Gamma$  and  $f : \mathcal{A} \rightarrow \mathcal{B}$ , naturality means

$$f_{\Delta} \circ \mathcal{A}_{\tau} = \mathcal{B}_{\tau} \circ f_{\Gamma}.$$

In words, first weakening with  $\tau$  and then applying  $f$  has the same effect as applying  $f$  first and perform the weakening later. This guarantees some form of *parametricity* of  $f$  in the context, in particular,  $f$  cannot make decisions based on the length of the context or its precise contents.

Presheaves over OPE and natural transformations form a category PSh, traditionally called  $\widehat{\text{OPE}}$ .

**Exercise 3** *Prove that PSh is indeed a category.*

**Exercise 4** *Prove that the following functions are morphisms in PSh:*

$$\begin{array}{llll} \text{hyp} & : & \text{Hyp}(A) & \rightarrow \text{Ne}(A) \\ \text{andE}_1 & : & \text{Ne}(A \wedge B) & \rightarrow \text{Ne}(A) \\ \text{orl}_1 & : & \text{Nf}(A) & \rightarrow \text{Nf}(A \vee B) \\ \text{falseE} & : & \text{Ne}(\perp) & \rightarrow \text{Nf}(C) \end{array}$$

*Find other examples of presheaf morphisms.*

## 4 NbE for the negative fragment

We wish to define a normalization function for derivations of any formula  $A$

$$\text{norm}^A : \text{Tm}(A) \rightarrow \text{Nf}(A).$$

Normalization by evaluation (NbE) achieves normalization by first evaluating terms, obtaining values and functions that compute by means of the computation in the meta language, i. e., in SET. As a second step, these values are reified back into syntax, yielding a normal form. Technically, we define a suitable presheaf  $\llbracket A \rrbracket$  for each formula  $A$  and two functions

$$\begin{array}{llll} \text{eval}^A & : & \text{Tm}(A) & \rightarrow \llbracket A \rrbracket \\ \text{reify}^A & : & \llbracket A \rrbracket & \rightarrow \text{Nf}(A) \end{array}$$

which compose to  $\text{norm}$ .

### 4.1 Semantics, reflection and reification

For now, let us focus on the negative formulas  $\top$ ,  $A \wedge B$ ,  $A \Rightarrow B$  and atoms  $P$ .

$$\begin{array}{llll} \llbracket P \rrbracket & = & \text{Nf}(P) & \text{reify}_{\Gamma}^P t & = & t \\ \llbracket \top \rrbracket & = & \hat{1} & \text{reify}_{\Gamma}^{\top} - & = & \text{trueI} \\ \llbracket A \wedge B \rrbracket & = & \llbracket A \rrbracket \hat{\times} \llbracket B \rrbracket & \text{reify}_{\Gamma}^{A \wedge B} (a, b) & = & \text{andI} (\text{reify}_{\Gamma}^A a) (\text{reify}_{\Gamma}^B b) \end{array}$$

Herein, we use the unit presheaf  $\hat{1}_\Gamma = 1$  where 1 is the unit set, and the pointwise product of presheaves  $(\mathcal{A} \hat{\times} \mathcal{B})_\Gamma = \mathcal{A}_\Gamma \times \mathcal{B}_\Gamma$ . We note that by these definitions, the category PSh has products, with the pointwise terminal morphisms:

$$\begin{aligned} ! & : \mathcal{A} \rightarrow \hat{1} & [-, \_] & : (f : \mathcal{C} \rightarrow \mathcal{A})(g : \mathcal{C} \rightarrow \mathcal{B}) \rightarrow (\mathcal{C} \rightarrow \mathcal{A} \hat{\times} \mathcal{B}) \\ !_\Gamma(a) & = () & [f, g]_\Gamma(c) & = (f(c), g(c)) \end{aligned}$$

The interpretation  $\llbracket A \Rightarrow B \rrbracket$  of implication will follow the Brouwer-Heyting-Kolmogorov (BHK) interpretation of intuitionistic logic:

*A proof of  $A \Rightarrow B$  is a method turning any proof of  $A$  into a proof of  $B$ .*

However, the direct lifting of the function space  $\llbracket A \Rightarrow B \rrbracket_\Gamma = \llbracket A \rrbracket_\Gamma \rightarrow \llbracket B \rrbracket_\Gamma$  does not work. It does not give a presheaf because of the contravariant occurrence of presheaf  $\llbracket A \rrbracket$ . Instead, we force monotonicity by quantifying over all OPEs of  $\Gamma$ :

$$\llbracket A \Rightarrow B \rrbracket_\Gamma = \prod_{\Delta: \text{Cxt } \tau: \Delta \leq \Gamma} (\llbracket A \rrbracket_\Delta \rightarrow \llbracket B \rrbracket_\Delta)$$

Reification is also challenging. Let  $f : \llbracket A \Rightarrow B \rrbracket_\Gamma$  and attempt

$$\text{reify}_\Gamma^{A \Rightarrow B} f = \text{impl} (\text{reify}_{\Gamma.A}^B (f_{\Gamma.A} (\text{weak}_A \text{id}_\Gamma) ?))$$

where hole “?” should be filled with an element of  $\llbracket B \rrbracket_{\Gamma.A}$  which intuitively should represent the new hypothesis  $\text{top} : \text{Hyp}(A)_{\Gamma.A}$  introduced by  $\text{impl}$ . Thus we need a natural transformation  $\text{Hyp}(A) \rightarrow \llbracket A \rrbracket$  which *reflects* variables into the semantics. In order to define it by induction on  $A$ , we need to generalize it to

$$\text{reflect}^A : \text{Ne}(A) \rightarrow \llbracket A \rrbracket.$$

Let us make sure we can define it at atoms and conjunctions:

$$\begin{aligned} \text{reflect}^P & = \text{ne} & : \text{Ne}(P) \rightarrow \text{Nf}(P) \\ \text{reflect}^\top & = ! & : \text{Ne}(\top) \rightarrow \hat{1} \\ \text{reflect}^{A \wedge B} & = \langle \text{reflect}^A \circ \text{andE}_1, \text{reflect}^B \circ \text{andE}_2 \rangle & : \text{Ne}(A \wedge B) \rightarrow \llbracket A \rrbracket \hat{\times} \llbracket B \rrbracket \end{aligned}$$

Observe the eliminations  $\text{andE}_i$  introduced to reflect conjunctive hypotheses!

Now, we can complete the story for implication. Let  $\tau : \Delta \leq \Gamma$  and  $a : \llbracket A \rrbracket_\Delta$  and set:

$$\begin{aligned} \text{reflect}^{A \Rightarrow B} & : \text{Ne}(A \Rightarrow B) \rightarrow \llbracket A \Rightarrow B \rrbracket \\ \text{reflect}_\Gamma^{A \Rightarrow B} t \Delta \tau a & = \text{reflect}_\Delta^B (\text{impE} (\text{Ne}(B)_\tau t) (\text{reify}_\Delta^A a)) \\ \text{reify}_\Gamma^{A \Rightarrow B} & : \llbracket A \Rightarrow B \rrbracket \rightarrow \text{Nf}(A \Rightarrow B) \\ \text{reify}_\Gamma^{A \Rightarrow B} f & = \text{impl} (\text{reify}_{\Gamma.A}^B (f_{\Gamma.A} (\text{weak}_A \text{id}_\Gamma) (\text{reflect}_{\Gamma.A}^A (\text{hyp top})))) \end{aligned}$$

**Exercise 5** Let  $A = ((P \Rightarrow Q) \wedge (Q \Rightarrow \top)) \Rightarrow Q$  for some atoms  $P, Q$ . Compute  $\text{reflect}_A^A (\text{hyp top})$ .

**Exercise 6** Let  $A = (P \Rightarrow Q) \Rightarrow (P \Rightarrow Q)$  for some atoms  $P, Q$ . Compute  $\text{reify}_A^A \circ \text{reflect}_A^A$ .

**Exercise 7** Using the constructions in this section, show that the category of presheaves and natural transformations is Cartesian closed.

## 4.2 Evaluation

Evaluation is turning an expression  $t : \mathsf{Tm}(A)_\Gamma$  into a value in  $\llbracket A \rrbracket$ . However, a direct definition  $\mathsf{eval}^A : \mathsf{Tm}(A) \rightarrow \llbracket A \rrbracket$  by induction on the expression fails in case `impl`:

$$\begin{aligned} \mathsf{eval}_\Gamma^{A \Rightarrow B} & : \mathsf{Tm}(A)_\Gamma \rightarrow \llbracket A \Rightarrow B \rrbracket_\Gamma \\ \mathsf{eval}_\Gamma^{A \Rightarrow B} (\mathsf{impl} \ t) \ \Delta \ a & = ? \end{aligned}$$

To be able to pass the argument  $a$  to the function  $\mathsf{eval}(\mathsf{impl} \ t)$ , we have to generalize evaluation to take an valuation of the context into account.

$$\mathsf{eval}_\Gamma^{A \Rightarrow B}(t) : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \Rightarrow B \rrbracket$$

Here,  $\llbracket \Gamma \rrbracket$  is the presheaf obtained as the product of the hypotheses contained in  $\Gamma$ :

$$\begin{aligned} \llbracket \emptyset \rrbracket & = \hat{1} \\ \llbracket \Gamma.A \rrbracket & = \llbracket \Gamma \rrbracket \hat{\times} \llbracket A \rrbracket \end{aligned}$$

Looking up values in the environment is the following family of natural transformations:

$$\begin{aligned} \mathsf{lookup}_\Gamma^A & : \mathsf{Hyp}(A)_\Gamma \rightarrow (\llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket) \\ \mathsf{lookup}_{\Gamma.A}^A(\mathsf{top}) \ \Delta \ (\gamma, a) & = a \\ \mathsf{lookup}_{\Gamma.B}^A(\mathsf{pop} \ x) \ \Delta \ (\gamma, a) & = \mathsf{lookup}_\Gamma^A(x) \ \Delta \ \gamma \end{aligned}$$

**Exercise 8** Express lookup via the projections  $\pi_i : \mathcal{A}_1 \hat{\times} \mathcal{A}_2 \rightarrow \mathcal{A}_i$  out of the presheaf product.

We are ready to define evaluation:

$$\begin{aligned} \mathsf{eval}_\Gamma^A(\mathsf{hyp} \ x) & = \mathsf{lookup}_\Gamma^A(x) \\ \mathsf{eval}_\Gamma^{A \Rightarrow B}(\mathsf{impl} \ t) \ \Delta \ \gamma \ \Delta' \ (\tau : \Delta' \leq \Delta) \ a & = \mathsf{eval}_{\Gamma.A}^B(t) \ \Delta' \ (\llbracket \Gamma \rrbracket_\tau \gamma, a) \\ \mathsf{eval}_\Gamma^B(\mathsf{impE} \ t \ u) \ \Delta \ \gamma & = \mathsf{eval}_\Gamma^{A \Rightarrow B}(t) \ \Delta \ \gamma \ \Delta \ \mathsf{id}_\Delta \ (\mathsf{eval}_\Gamma^A(u) \ \Delta \ \gamma) \end{aligned}$$

**Exercise 9** Complete the definition of eval for conjunctions.

**Exercise 10** Express eval in terms of the operations of the CCC of presheaves!

## 4.3 Normalization

Normalization is reification after evaluation in the identity environment:

$$\begin{aligned} \mathsf{norm}^A & : \mathsf{Tm}(A) \rightarrow \mathsf{Nf}(A) \\ \mathsf{norm}_\Gamma^A t & = \mathsf{reify}_\Gamma^A(\mathsf{eval}_\Gamma^A(t) \ \Gamma \ \mathsf{env}_\Gamma) \end{aligned}$$

The identity environment  $\mathsf{env}_\Gamma$  is defined by induction on  $\Gamma$  as follows:

$$\begin{aligned} \mathsf{env}_\Gamma & : \llbracket \Gamma \rrbracket_\Gamma \\ \mathsf{env}_\emptyset & = () \\ \mathsf{env}_{\Gamma.A} & = (\llbracket \Gamma \rrbracket_{\mathsf{weak}_A \ \mathsf{id}_\Gamma}(\mathsf{env}_\Gamma), \mathsf{reflect}_{\Gamma.A}^A(\mathsf{hyp} \ \mathsf{top})) \end{aligned}$$

#### 4.4 Digression: The CCC of reifiable presheaves

Instead of defining the semantics  $\llbracket A \rrbracket$  of formula  $A$  by induction on  $A$ , we can consider a category whose objects are interpretations of formulas, and show that this category is Cartesian closed.

An object in this category would be a quadruple consisting of:

$\mathcal{A}$	:	PSh	a presheaf
$A$	:	Form	a formula
$\text{reflect}^A$	:	$\text{Ne}(A) \rightarrow \mathcal{A}$	a method to reflect neutrals into $\mathcal{A}$
$\text{reify}^A$	:	$\mathcal{A} \rightarrow \text{Nf}(A)$	a method to reify values of $\mathcal{A}$

**Exercise 11** Complete this definition by a suitable notion of morphism.

**Exercise 12** Show that this category is Cartesian closed.

### 5 Disjunction and absurdity

For sets  $S_1, S_2 : \text{SET}$  let  $S_1 + S_2$  denote the disjoint union with injections  $\text{inj}_i : S_i \rightarrow S_1 + S_2$  and case distinction distinction  $[f_1, f_2] : S_1 + S_2 \rightarrow T$  for  $f_i : S_i \rightarrow T$  such that  $[f_1, f_2] \circ \text{inj}_i = f_i$ .

Presheaf coproduct is given pointwise by  $(\mathcal{A} \hat{+} \mathcal{B})_\Gamma = \mathcal{A}_\Gamma + \mathcal{B}_\Gamma$ .

**Exercise 13** Show that  $\hat{+}$  is indeed a coproduct in PSh.

#### 5.1 Semantics, reflection and reification

The first idea how to model disjunction,  $\llbracket A \vee B \rrbracket_\Gamma = \llbracket A \rrbracket_\Gamma + \llbracket B \rrbracket_\Gamma$ , fails because we cannot reflect disjunctive hypotheses. E. g., an element of  $\llbracket A \vee B \rrbracket_{A \vee B}$  should be of the form  $\text{inj}_1 a$  or  $\text{inj}_2 b$ , but the decision on first or second injection cannot be made here; it depends on the hypothesis  $A \vee B$ . The solution is to extend the semantics to allow case distinction on variables (and neutrals) before making the decisions needed for disjunctions.

Given a presheaf  $\mathcal{A}$ , we define a new presheaf  $\text{Cover}(\mathcal{A})$  inductively by the following constructors:

$$\frac{a : \mathcal{A}_\Gamma}{\text{returnC } a : \text{Cover}(\mathcal{A})_\Gamma} \quad \frac{t : \text{Ne}(\perp)_\Gamma}{\text{falseC } t : \text{Cover}(\mathcal{A})_\Gamma}$$

$$\frac{t : \text{Ne}(C \vee D)_\Gamma \quad c : \text{Cover}(\mathcal{A})_{\Gamma.C} \quad d : \text{Cover}(\mathcal{A})_{\Gamma.D}}{\text{orC } t \ c \ d : \text{Cover}(\mathcal{A})_\Gamma}$$

**Exercise 14** Show that  $\text{Cover}(\mathcal{A})$  is indeed a presheaf.

We can think of  $c : \text{Cover}(\mathcal{A})_\Gamma$  as a decision tree with leaves in  $\mathcal{A}$  and nodes labeled by terms we case on. Nodes of type **falseC** have no subtrees since *ex falsum quod libet*, and **orC** nodes casing on  $t : \text{Ne}(C \vee D)_\Gamma$  have two subtrees, one having a new hypothesis  $C$ , and one having a new hypothesis  $D$ .

We can replace all the leaves according to a function  $f : \mathcal{A} \rightarrow \mathcal{B}$ , this makes  $\text{Cover}$  a *functor* in the category of presheaves. We can also replace all the leaves by new case trees via a natural transformation

$$\text{joinC} : \text{Cover}(\text{Cover}(\mathcal{A})) \rightarrow \text{Cover}(\mathcal{A})$$

which makes  $\text{Cover}$  a *monad* in  $\text{PSh}$ .

**Exercise 15** Define the functorial action  $\text{Cover}(f)$  and  $\text{joinC}$  and prove the monad laws.

Further, if the leaves of a case tree are normal forms, we can turn it into one big normal form via:

$$\text{pasteNf} : \text{Cover}(\text{Nf}(A)) \rightarrow \text{Nf}(A)$$

**Exercise 16** Define  $\text{pasteNf}$ .

We have everything now to model disjunction:

$$\begin{aligned} \llbracket A \vee B \rrbracket &= \text{Cover}(\llbracket A \rrbracket \hat{+} \llbracket B \rrbracket) \\ \text{reflect}^{A \vee B} &: \text{Ne}(A \vee B) \rightarrow \llbracket A \vee B \rrbracket \\ \text{reflect}_{\Gamma}^{A \vee B} t &= \text{orC } t \text{ (returnC (inj}_1\text{(reflect}_{\Gamma,A}^A \text{(hyp top))}))} \\ &\quad \text{(returnC (inj}_2\text{(reflect}_{\Gamma,B}^B \text{(hyp top))}))} \\ \text{reifyOr}^{A \vee B} &: \llbracket A \rrbracket \hat{+} \llbracket B \rrbracket \rightarrow \text{Nf}(A \vee B) \\ \text{reifyOr}_{\Gamma}^{A \vee B}(\text{inj}_1 a) &= \text{orl}_1(\text{reify}_{\Gamma}^A a) \\ \text{reifyOr}_{\Gamma}^{A \vee B}(\text{inj}_2 b) &= \text{orl}_2(\text{reify}_{\Gamma}^B b) \\ \text{reify}^{A \vee B} &: \llbracket A \vee B \rrbracket \rightarrow \text{Nf}(A \vee B) \\ \text{reify}^{A \vee B} &= \text{pasteNf} \circ \text{Cover}(\text{reifyOr}^{A \vee B}) \end{aligned}$$

Reification works by first reifying the leaves of the case tree using  $\text{reifyOr}$ , and then turning the whole tree into a normal form using  $\text{pasteNf}$ .

**Exercise 17** Compute  $\text{reify}_{P \vee Q}^{P \vee Q}(\text{reflect}_{P \vee Q}^{P \vee Q}(\text{hyp top}))$ .

For absurdity, we use the presheaf  $\hat{0}_{\Gamma} = \{\}$  with initial morphism  $\perp\text{-elim} : \hat{0} \rightarrow \mathcal{C}$ .

$$\begin{aligned} \llbracket \perp \rrbracket &= \text{Cover } \hat{0} \\ \text{reflect}^{\perp} &: \text{Ne}(\perp) \rightarrow \llbracket \perp \rrbracket \\ \text{reflect}_{\Gamma}^{\perp} t &= \text{falseC } t \\ \text{reify}^{\perp} &: \llbracket \perp \rrbracket \rightarrow \text{Nf}(\perp) \\ \text{reify}^{\perp} &= \text{pasteNf} \circ \text{Cover}(\perp\text{-elim}) \end{aligned}$$

To reify a value  $c : \llbracket \perp \rrbracket$ , observe that because  $\hat{0}$  is empty,  $c$  can only be a case tree without leaves, i. e., all branches end in  $\text{falseC}$  nodes. Thus, the mapping  $\text{Cover}(\perp\text{-elim}) : \text{Cover}(\hat{0}) \rightarrow \text{Cover}(\text{Nf}(\perp))$  is merely a type cast. The subsequent  $\text{pasteNf}$  will turn the leafless tree  $c$  into a normal form consisting entirely of splits  $\text{orE}$  and  $\text{falseE}$ .

**Exercise 18** Prove that  $\text{reify}^A \circ \text{reflect}^A$  produces the long  $\eta$ -normal form if applied to variables (i. e., hypotheses).



## 5.2 Evaluation

Evaluation poses one more challenge. Consider the term:

$$\frac{x : \text{Hyp}(C \vee D)_\Gamma \quad f : \text{Tm}(A \Rightarrow B)_{\Gamma.C} \quad g : \text{Tm}(B)_{\Gamma.D.A} \quad a : \text{Tm}(A)_\Gamma}{\text{impE}(\text{orE}(\text{hyp } x) f (\text{impl } g)) a : \text{Tm}(B)_\Gamma}$$

There is a redex  $\text{impE}(\text{impl } g) a$  separated by an  $\text{orE}$ . The function term  $\text{orE}(\text{hyp } x) f (\text{impl } g)$  would naturally evaluate to a case tree with functions in the leaves. To apply it to the evaluation of  $a$ , we have to apply each leaf to that argument. We facilitate this by implementing

$$\text{paste}^{A \Rightarrow B} : \text{Cover}(\llbracket A \Rightarrow B \rrbracket) \dot{\rightarrow} \llbracket A \Rightarrow B \rrbracket$$

that turns a case tree of functions into a function which expects an argument that will then be passed to all the functions in the case tree. In fact, we need pasting at all formulas  $A$ :

$$\begin{aligned} \text{paste}^A & : \text{Cover} \llbracket A \rrbracket \dot{\rightarrow} \llbracket A \rrbracket \\ \text{paste}^P & = \text{pasteNf} \\ \text{paste}^\perp & = \text{joinC} \\ \text{paste}^{A \vee B} & = \text{joinC} \\ \text{paste}^{A \wedge B} & = \langle \text{paste}^A \circ \text{Cover}(\pi_1), \text{paste}^B \circ \text{Cover}(\pi_2) \rangle \\ \text{paste}_{\Gamma}^{A \Rightarrow B} c_\Delta (\tau : \Delta \leq \Gamma) (a : \llbracket A \rrbracket_\Delta) & = (\text{paste}_\Delta^B \circ \text{mapC}_\Delta(\varphi) \circ \text{Cover}(\llbracket A \Rightarrow B \rrbracket_\tau)) c \end{aligned}$$

In the case of implication  $A \Rightarrow B$ , we need a stronger version of functoriality of  $\text{Cover}$  which only requires a

$$\varphi : \prod_{\Phi} \prod_{\delta: \Phi \leq \Delta} \mathcal{A}_\Phi \rightarrow \mathcal{B}_\Phi$$

to produce a  $\text{mapC}_\Delta(\varphi) : \text{Cover}(\mathcal{A})_\Delta \rightarrow \text{Cover}(\mathcal{B})_\Delta$ . In our case,  $\mathcal{A} = \llbracket A \Rightarrow B \rrbracket$  and  $\mathcal{B} = \llbracket B \rrbracket$  and

$$\varphi_\Phi \delta f = f \text{id}_\Phi(\llbracket A \rrbracket_\delta a).$$

Pasting allows us to complete the definition of evaluation.

$$\begin{aligned} \text{eval}_\Gamma^A & : \text{Tm}(A)_\Gamma \rightarrow \llbracket \Gamma \rrbracket \dot{\rightarrow} \llbracket A \rrbracket \\ \text{eval}_\Gamma^C(\text{orE } t u v)_\Delta \gamma & = \text{paste}_\Delta^C(\text{mapC}_\Delta(\lambda_\Phi (\tau : \Phi \leq \Delta) \rightarrow [f, g]) c) \\ \text{where } c & = \text{eval}_\Gamma^{A \vee B}(t)_\Delta \gamma \\ f a & = \text{eval}_{\Gamma.A}^C(u)_\Phi(\llbracket \Gamma \rrbracket_\tau \gamma, a) \\ g b & = \text{eval}_{\Gamma.B}^C(v)_\Phi(\llbracket \Gamma \rrbracket_\tau \gamma, b) \end{aligned}$$

**Exercise 19** *Really complete the definition of eval (all missing cases).*

That's it!

## 6 Literature

Further reading on normalization by evaluation in general:

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4. A. Abel. *Normalization by Evaluation: Dependent Types and Impredicativity*. Habilitation thesis, Ludwig-Maximilians-University Munich, 2013

Literature dealing in particular with the case of disjunction / disjoint sums:

1. O. Danvy. Type-directed partial evaluation. In *POPL'96*, 1996
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6. F. Barral. *Decidability for non-standard conversions in lambda-calculus*. PhD thesis, Ludwig-Maximilians-University Munich, 2008
7. G. Scherer. Deciding equivalence with sums and the empty type. In *POPL'17*, 2017

## 7 Conclusion

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